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Perturbative Methods for Modelling

(Méthodes perturbatives pour la modélisation)

$y(x) = \sum_{n=0}^{\infty} a_n\, x^n$

$\epsilon\, y'' + (1+\epsilon)\, y' + y = 0$

$y'' + \frac{1}{x}\, y' - \left(1 + \frac{\nu^2}{x^2}\right) y = 0$

$K_\nu(x) \sim \sqrt{\frac{\pi}{2\,x}}\, e^{-x}$ for $x \to +\infty$

$L\, y = y'' + \frac{p(x)}{x}\, y' + \frac{q(x)}{x^2} = 0$

$J_\nu = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-x^2/4)^n}{n!\,\Gamma(\nu+n+1)}$

$y'' - y = 1/\cosh x$ with $y(\pm\infty) = 0$

Duffing equation: $y'' + y + \epsilon\, y^3 = 0$

multiscale analysis $y(t) \sim Y_0(t, \epsilon t)$ for $\epsilon \to 0$

$y(x) \sim c_\pm\, x^{-1/4}\, e^{\pm 2\, x^{3/2}/3} \left(1 \pm \frac{5}{48}\, x^{-3/2}\right)$

Stirling's formula $\Gamma(x) \sim x^x\, e^{-x} \sqrt{\frac{2\pi}{x}}$ for $x \to +\infty$

Modified Bessel equation: $x^2\, y'' + x\, y' - (x^2 + \nu^2)\, y = 0$

Bessel equation: $x^2\, y'' + x\, y' + (x^2 - \nu^2)\, y = 0$

Rayleigh oscillator: $y'' + y = \epsilon\, \left[y' - \frac{1}{3}\,(y')^3\right]$

$\int_{-\infty}^{\infty} \cos(x\, t^2 - t) dt \sim \sqrt{\frac{\pi}{2\,x}}$ for $x \to +\infty$

$K_0(x) = e^{-x} \int_0^{\infty} (t^2 + 2\,t)^{-1/2}\, e^{-x\,t}\, dt$

Schrödinger equation $\epsilon^2\, y'' = Q(x)\, y$

$I_\nu(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\, x\right)^{2\,n+\nu}}{n!\,\Gamma(\nu+n+1)}$

$y_{unif} = y_{in} + y_{out} + y_{match}$

Airy equation: $y'' = x\, y$

$y'' - 3\, y' + 2\, y = e^{4\,x}$

$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt$

Master 2 Recherche DET

Dynamique des Fluides, Énergétique et Transferts

Université Fédérale Toulouse Midi-Pyrénées (UFTMiP)

Année 2016-2017, version du November 19, 2016

Syllabus

Français : Méthodes perturbatives pour la modélisation

La modélisation repose sur une palette d'outils de mathématiques appliquées. Le livre "Advanced mathematical methods for scientists and engineers", de Carl M. Bender et Steven A. Orszag ([1]), présente des méthodes asymptotiques très courantes en physique. Le cours est centré sur une lecture active de cet ouvrage en se concentrant sur un choix de chapitres particulièrement pertinents pour la mécanique des fluides. Une approche des fonctions spéciales (Airy, Bessel, ...) est effectuée à l'aide des séries de Frobenius, solutions d'équations différentielles linéaires. Les méthodes de la phase stationnaire et du col, très présentes dans l'étude des ondes et des instabilités, permettent de décrire le comportement asymptotique d'intégrales à grand paramètres. Enfin, trois méthodes asymptotiques sont incontournables dans de nombreux problèmes de recherche : analyse de couche limite, théorie WKB et méthode des échelles multiples. Au-delà de ces objectifs principaux du cours, assimilés au moyen de nombreux exercices, une sensibilisation aux contenus des autres chapitres du livre est visée.

English: Perturbation methods for modelling

Modelling is based on a range of applied mathematics tools. The book "Advanced mathematical methods for scientists and engineers", by Carl M. Bender and Steven A. Orszag ([1]), presents very common asymptotic methods in physics. The course focuses on an active reading of this book focusing on a selection of chapters particularly relevant for fluid mechanics. An approach of special functions (Airy, Bessel, ...) is performed using a Frobenius series, solutions of linear differential equations. The methods of the stationary phase and steepest descent, very present in the study of waves and instabilities, can describe the asymptotic behaviour of integrals with large parameters. Finally, three asymptotic methods are unavoidable in many research problems: boundary layer analysis, WKB theory and multiple scales methods. Beyond these main objectives of the course, assimilated through many exercises, an outreach to the contents of other chapters of the book is targeted.

References

[1] Carl M. Bender et Steven A. Orszag, Advanced mathematical methods for scientists and engineers, McGraw-Hill 1978

[2] `http://mooc.inp-toulouse.fr/` → `Mathématiques` → `Perturbative Methods`

Chapter 1

Exercices for the oral presentation

The goal of this course is to become familiar with the book "Advanced mathematical methods for scientists and engineers" ([1]), which content can be useful for a wide variety of research subjects, more particularly in the domain of Fluid Mechanics (S. Orszag has be a pioneer in Fluid Mechanics and C. Bender has worked on quantum mechanics). The subtitle of the book, "Asymptotic Methods and Perturbation Theory", indicates that the focus is made on applications.

Teaching the content of the whole book would require a large amount of hours. Here, a guided tour inside the book is organized through a series of chosen exercices. These exercices can reasonably be taught during a 15 hour class. The exercices with the tag "optional" can be added for a very intensive teaching or for a 20 hour class.

The answers of these exercices are to be found, most of the time, in the reference book ([1]). In a few cases, solutions, hints or complementary explanations are proposed.

Physical applications are briefly evocated in some places to motivated the course.

Readers are encouraged to register to the following pedagogical plaform (password: masterdet) and create their own account (connexion → autres utilisateurs → Première visite):

`http://mooc.inp-toulouse.fr/` → `Mathématiques` → `Perturbative Methods`

They will find guidance to solve the exercices or deposit contributions. They are encouraged to solve exercices taken from the book and describe them, with hints about the solution, on the pedagogical platform.

1.1 Ordinary Differential Equations

1.1.1 Variation of parameters

Example 3, p. 15

Solve the following equation with the parameter variation method: $y'' - 3\,y' + 2\,y = e^{4\,x}$.

Answer

The general solution is $y(x) = c_1\,e^{x} + c_2\,e^{2\,x} + \frac{1}{6}\,e^{4\,x}$ where c_1 and c_2 are constants.

Method

The solution of the homogeneous equation is $y_h(x) = u_1\,y_1(x) + u_2\,y_2(x)$ with $y_1 = \exp(x)$ and

$y_2 = \exp(2\,x)$, where u_1 and u_2 are constants.

We look at solutions $y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$ such that:

$$u_1'(x)\,y_1(x) + u_2'(x)\,y_2(x) = 0\;. \tag{1.1}$$

Since $L\,y_1 = L\,y_2 = 0$ and $L\,y = f(x)$ with $L = \frac{d^2}{dx^2} - 3\,\frac{d}{dx} + 2$ and $f(x) = \exp(4\,x)$, we have:

$$u_1'(x)\,y_1'(x) + u_2'(x)\,y_2'(x) = f(x)\;. \tag{1.2}$$

We can then compute:

$$u_1' = -f\,y_2/W \qquad \text{and} \qquad u_2' = f\,y_1/W \qquad \text{with} \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\;. \tag{1.3}$$

Replacing y_1, y_2 and f with their expression, we get $W = \exp(3\,x)$, $u_1' = -\exp(3\,x)$ and $u_2' = \exp(2\,x)$, leading to the solution.

1.1.2 Green function

Example 5, p. 19

Solve with the Green function method the equation $y'' - y = 1/\cosh x$ with $y(\pm\infty) = 0$.

Answer

The Green function is $G(x,a) = \frac{1}{2}\,e^{-|x-a|}$. The general solution:

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-a|}\,f(a)\,da\;.$$

reads

$$y(x) = -e^{x}\,\ln\sqrt{e^{-2x}+1} - e^{-x}\,\ln\sqrt{e^{2x}+1}\;.$$

One can prove that $y(\pm\infty) = 0$. This result could have been found using the constants variation method.

Physical applications

The Green function of the Laplace equation is defined by $\Delta G = \delta(\underline{x})$ with G bounded at infinity. It reads $G = 1/(4\,\pi\,\|\underline{x}\|)$ for $\underline{x} \in I\!R^3$. It explains the Biot and Savart law for electrostatics as well as many other physical applications. It reads $G = -\mathrm{Ln}\,\|\underline{x}\|/(2\,\pi)$ for $\underline{x} \in I\!R^2$.

The Green function of the Helmoltz equation is defined by $\Delta G + k^2 G = \delta(\underline{x})$ with G bounded at infinity. It reads $G = e^{i\,k\,\|\underline{x}\|}/(4\,\pi\,\|\underline{x}\|)$ for $\underline{x} \in I\!R^3$ and $G = e^{i\,k\,|x|}/(2\,i\,k)$ for $x \in I\!R$. It is widedly used for waves applications, where k is the wave number.

1.2 Difference Equations

1.2.1 Gamma function

Example 2, p. 38, Figure 2.1, p. 39, Exercice 2.6, p. 54,

The Gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t}\,t^{z-1}\,dt$ for $\mathrm{Re}(z) > 0$.

- Show that $\Gamma(1) = \Gamma(2) = 1$
- Show that $\Gamma(1/2) = \sqrt{\pi}$
- Show that $\Gamma(z+1) = z\,\Gamma(z)$

The Gamma function is extended to the complex plan with this last relation.

- Show that $z = 0, -1, -2, ...$ are simple poles
- Compute the residues of $\Gamma(z)$ for these poles

1.3 Approximate Solution of Linear Differential Equations

1.3.1 Singular points

Exercice 3,3, p. 137

Classify all the singular points (finite and infinite) of the following equations:

- Airy equation: $y'' = x\,y$
- Bessel equation: $x^2\,y'' + x\,y' + (x^2 - \nu^2)\,y = 0$
- Hypergeometric equation: $x\,(1-x)\,y'' + [c - (a+b+1)\,x]\,y' - a\,b\,y = 0$
- Parabolic cylinder equation: $y'' + \left(\nu + \frac{1}{2} - \frac{1}{2}x^2\right)\,y = 0$
- Mathieu equation: $y'' + [h - 2\,\theta\,\cos(2\,x)]\,y = 0$

Answer

We set $t = 1/x$ for the point $x_0 = \infty$ leading to $\frac{d}{dx} = -t^2\,\frac{d}{dt}$ and $\frac{d^2}{dx^2} = -t^4\,\frac{d^2}{dt^2} + 2\,t^3\,\frac{d}{dt}$.

Airy equation: all the finite points are regular. Since $g(t) = y(1/x)$ satisfies $g'' + 2\,g'/t^2 - g/t^3 = 0$, $x_0 = \pm\infty$ are irregular singular points.

Bessel equation: the point $x_0 = 0$ is a regular singular point. All the other finite points are regular. The points $x_0 = \pm\infty$ are irregular singular points.

Hypergeometric equation: the points $x_0 = 0$ and $x_1 = 1$ are regular singular points. All the other finite points are regular. The points $x_0 = \pm infty$ are irregular singular points.

Parabolic cylinder and Mathieu equations: all the finite points are regular. The points $x_0 = \pm\infty$ are irregular singular points.

Physical applications

The Airy equation describe the transition between oscillatory and evanescent waves.

Looking for solution of the Helmoltz equation $\Delta u + u = 0$ in 2D or 3D lead to the Bessel equations. In the 2D case, these solution described, for instance, the oscillation modes of a drum.

The parabolic cylinder equation are derived from the Schrödinger equation for a particule in a harmonic potential.

The Mathieu equation describe the dynamics of and oscillator forced by a periodic signal and is used to describe parmaetric instabilites.

1.3.2 Airy functions

Example 2, p. 67, Figure 3.1, p. 69

Compute the series $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$ solutions of the Airy equations $y'' = x\, y$.

1.3.3 Frobenius method

Section 3.3, p. 68

Describe the Frobenius method for

$$L\, y = y'' + \frac{p(x)}{x}\, y' + \frac{q(x)}{x^2} = 0\,, \tag{1.4}$$

where $p(x) = \sum_{n=0}^{\infty} p_n\, x^n$ and where $q(x) = \sum_{n=0}^{\infty} q_n\, x^n$ are analytic functions in a vicinity of the regular singular point $x = 0$.

Apply the Frobenius method for the modified Bessel equation :

$$y'' + \frac{1}{x}\, y' - \left(1 + \frac{\nu^2}{x^2}\right) y = 0\ , \tag{1.5}$$

where $\nu \in \mathbb{R}$. Consider the four cases:
1) $2\,\nu \notin \mathbb{N}$, 2) $\nu = \frac{1}{2} + N$ with $N \in \mathbb{N}$, 3) $\nu = 0$ and 4) $\nu = 1$.

We denote the modified Bessel function $I_\nu(x)$ by:

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\, x\right)^{2\,n+\nu}}{n!\, \Gamma(\nu + n + 1)}\ . \tag{1.6}$$

Description of the Frobenius method cases

We consider Frobenius series $y(x) = x^\alpha\, \sum_{n=0}^{\infty} a_n\, x^n$ and compute

$$L\, y(x, \alpha) = x^\alpha\, a_0\, P(\alpha) + x^\alpha \sum_{n=1}^{\infty} \left[a_n\, P(\alpha + n) + f_n\right] \tag{1.7}$$

with $P(\alpha) = \alpha^2 + (p_0 - 1)\, \alpha + q_0$ and $f_n = \sum_{k=0}^{n-1} \left[(\alpha + k)\, p_{n-k} + q_{n-k}\right] a_k$. We denotes by $\alpha_1 \leq \alpha_2$ the two roots of $P(\alpha)$.

I	$\alpha_1 - \alpha_2 \notin \mathbb{N}$				
II	$\alpha_1 - \alpha_2 = N \in \mathbb{N}$				
		II(a)	$N = 0,\ \alpha_1 = \alpha_2$		
		II(b)	$N \neq 0,\ \alpha_1 = \alpha_2 + N$		
				II(b)(i)	$f_N \neq 0$
				II(b)(ii)	$f_N = 0$

Table 1.1: Discussion for the Frobenius methods where $\alpha_1 \leq \alpha_2$ are the two roots of $P(\alpha)$.

If $y(x, \alpha) = x^\alpha\, \sum_{n=0}^{\infty} a_n(\alpha)\, x^n$, we note that $\frac{\partial}{\partial \alpha} y(x, \alpha) = y(x, \alpha)\, \ln x + \sum_{n=0}^{\infty} \frac{\partial a_n}{\partial \alpha}(\alpha)\, x^n$.

Application to the modified Bessel functions

1) $2\,\nu \notin \mathbb{N}$: case I of the Frobenius method. $I_\nu(x)$ and $I_{-\nu}(x)$ are the basic solutions.

2) $\nu = \frac{1}{2} + N$ with $N \in \mathbb{N}$: case II(b)(ii) of the Frobenius method. $I_\nu(x)$ and $I_{-\nu}(x)$ are the basic solutions.

3) $\nu = 0$: case II(a) of the Frobenius method. In addition to $I_0(x)$, one uses $K_0(x)$ defined by:

$$K_0(x) = -\left[\ln\left(\frac{1}{2}x\right) + \gamma\right] I_0(x) + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\,x\right)^{2\,n}}{(n!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right)\,, \tag{1.8}$$

where $\gamma = 0.5772...$ is the Euler's constant.

4) $\nu = 1$. Case I of the Frobenius method. In addition to $I_1(x)$, one uses $K_1(x)$ defined by:

$$K_1(x) = \left[\ln\left(\frac{1}{2}x\right) + \gamma\right] I_1(x) + \frac{1}{x} - \frac{x}{4} - \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\,x\right)^{2\,n+1}}{n!\,(n+1)!} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + \frac{1}{2\,n+2}\right)\,. \tag{1.9}$$

Physical applications

Special function are useful tools in physics. They appear as solutions of ordinary differential equations. The Frobenius methods is a way to compute or define them with series.

1.3.4 Irregular singular points

Exercice 3.38, p. 140 and Example 5, p. 100

By setting $y(x) = e^{S(x)}$ or $y(x) = e^{S_a(x)+C(x)}$ where $S(x) \sim S_a(x)$, find the asymptotic behavior of $y(x)$ for $x \to +\infty$ for the following equations:

(a) Modified Bessel equation: $x^2\,y'' + x\,y' - (x^2 + \nu^2)\,y = 0$.

(b) Parabolic cylinder equation: $y'' + (\nu + \frac{1}{2} - \frac{1}{4}\,x^2)\,y = 0$.

(c) Airy equation: $y'' = x\,y$.

Algebra

If we denote by $y'' + a(x)\,y' + b(x)\,y = 0$ the differential equation and set $y = e^{S(x)}$, we obtain $S'' + (S')^2 + a\,S\; + b = 0$. In order to find the asymptotic behavior, we set $|S''| \ll (S')^2$.

If $S(x) \sim S_a(x)$ is the dominant asymptotic behavior of $S(x)$, we can set $y(x) = e^{S_a(x)+C(x)}$. By neglecting C'', one gets $S_a'' + (S_a' + C')^2 + a(S_a' + C') + b = 0$, which leads to the asymptotic behavior of C.

Solutions

(a) Modified Bessel equation, (3.5.7) p. 92: one gets $(S')^2 + \frac{1}{x}S' - \left(1 + \frac{\nu^2}{x^2}\right) = 0$ and thus $S' = -\frac{1}{2\,x} \pm \sqrt{1 + \frac{1+4\,\nu^2}{4\,x^2}} \sim -\frac{1}{2\,x} \pm 1$. One get $S(x) = -\frac{1}{2}\,\ln x \pm x + cst$ and thus:

$$y(x) \sim c_1\,x^{-1/2}\,e^{x} \text{ or } y(x) \sim c_2\,x^{-1/2}\,e^{-x} \text{ for } x \to \infty$$

(b) Parabolic cylinder equation, (3.5.11) p. 96: one gets $S(x) \sim S_a(x) = \pm\frac{1}{4}\,x^2$. By setting $y(x) = e^{\frac{1}{4}\,x^2} + C(x)$, we get $C' = -\frac{x}{2} + \sqrt{\frac{x^2}{4} - (\nu + 1)} \sim -(\nu+1)/4$, leading to $C(x) \sim \ln x^{-(\nu+1)}$. Using the same method with $y(x) = e^{-\frac{1}{4}\,x^2} + C(x)$, we finally get the two asymptotic behaviour:

$$y(x) \sim c_1\,x^{-\nu-1}\,e^{x^2/4} \text{ or } y(x) \sim c_2\,x^{\nu}\,e^{-x^2/4} \text{ for } x \to \infty.$$

(c) Airy equation, (3.5.16) p. 100: one gets $S(x) \sim S_a(x) = \pm\frac{2}{3}\,x^{3/2}$. By setting $y(x) = e^{\frac{2}{3}\,x^{3/2}} + C(x)$, we get $C' = -x^{1/2} - \epsilon\, x^{1/2}\sqrt{1-\frac{1}{2}\,x^{-3/2}}$ with $\epsilon = \pm 1$. Choosing $\epsilon = 1$ to avoid $C \sim S_a$, we get $C' \sim -\frac{1}{4\,x}$ and $C \sim \ln(x^{-1/4}) + cste$. Using the same method with $y(x) = e^{-\frac{2}{3}\,x^{3/2}} + C(x)$ but with $\epsilon = -1$, we finally get the two asymptotic behaviour:

$$y(x) \sim c_1\, x^{-1/4}\, e^{\frac{2}{3}\,x^{3/2}} \text{ or } y(x) \sim c_2\, x^{-1/4}\, e^{-\frac{2}{3}\,x^{3/2}} \text{ for } x \to \infty.$$

1.3.5 Bessel functions

Exercice 3.68, p. 143 and Example 3, p. 111

Show that the Bessel function

$$J_\nu = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-x^2/4)^n}{n!\,\Gamma(\nu+n+1)} \tag{1.10}$$

is solution of the Bessel equation $x^2\,y'' + x\,y' + (x^2-\nu^2)\,y = 0$.

Show that $J_\nu(x) = e^{-i\,\nu\,\pi/2}\,I_\nu\left(x\,e^{i\,\pi/2}\right)$.

Is it correct to write $y \sim c_1\,x^{-1/2}\,\cos\left(x - \frac{1}{2}\nu\,\pi - \frac{1}{4}\,\pi\right)$ and $y \sim c_2\,x^{-1/2}\,\sin\left(x - \frac{1}{2}\nu\,\pi - \frac{1}{4}\,\pi\right)$ for $x \to +\infty$? Why?

We denote by $Y_\nu(x)$ the solution of the Bessel equation whose graph "closely resembles" to the one of $(2/\pi)^{1/2}\,x^{-1/2}\,\sin\left(x - \frac{1}{2}\nu\,\pi - \frac{1}{4}\,\pi\right)$. Why is it unique?

1.4 Approximate Solution of Nonlinear Differential Equations

1.4.1 Critical points

Example 4, p. 179 and Figure 4.15, p. 180

Draw the trajectories of the Volterra equations $\dot{x}_1 = x_1 - x_1\,x_2$ and $\dot{x}_2 = -x_2 + x_1\,x_2$ in the vicinity of its critical points. Draw approximatively the entire phase portrait.

Physical applications

Many physical applications lead to ordinary differential equations for a low number of degrees of freedom. This is the case, for example, of the trajectory of particule in a fluid flow. Studying the equilibria and the trajectories in their vicinity is a powerful tools to have a hint of the complete dynamics.

1.5 Asymptotic Expansion of Integrals

1.5.1 Watson's Lemma

Example 3, p. 265

An integral representation of the modified Bessel function $K_0(x)$ is

$$K_0(x) = e^{-x}\int_0^\infty (t^2+2\,t)^{-1/2}\,e^{-x\,t}\,dt\,. \tag{1.11}$$

Using Watson's Lemma, show the following asymptotic expansion of $K_0(x)$ for $x \to +\infty$:

$$K_0(x) \sim e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{[\Gamma(n+1/2)]^2}{2^{n+1/2}\, n!\, \Gamma(1/2)\, x^{n+1/2}} \; . \tag{1.12}$$

1.5.2 Laplace's method

Example 6(g) and Example 10, p. 268 and p.275

The modified Bessel function $K_\nu(x)$ has the following integral representation for $x > 0$:

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu\, t)\, dt \; . \tag{1.13}$$

Using Laplace's method, show that $K_\nu(x) \sim \sqrt{\frac{\pi}{2\,x}}\, e^{-x}$ for $x \to +\infty$.

The Gamma function $\Gamma(x)$ has the following integral representation for $x > 0$:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \; . \tag{1.14}$$

Using Laplace's method, show the Stirling's formula $\Gamma(x) \sim x^x\, e^{-x} \sqrt{\frac{2\,\pi}{x}}$ for $x \to +\infty$.

Formula

We recall that $\int_a^b f(t)\, e^{x\,\phi(t)}\, dt \sim 2\, f(c)\, e^{x\,\phi(c)}\, [-x\, \phi^{(p)}(c)]^{-1/p}\, \Gamma(1/p)\, (p!)^{1/p}/p$ for $x \to \infty$ when $\phi(t)$ can be approximated, in the vicinity of it maximum, by $\phi(c) + \frac{1}{p!}\,(t-c)^p\, \phi^{(p)}(c)$ if it is reached for $c \in]a, b[$. Since $\phi(c)$ is a maximum, we have p even and $\phi^{(p)}(c) < 0$.

1.5.3 Path deformation in the complex plane

Example 2, p. 283 and Figure 6.6 p. 284

By deforming the integration path in the complex plane and using the Laplace's method, show that $I(x) = \int_0^\infty e^{i\,x\,t^2}\, dt \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{i\,\pi/4}$ for $x \to +\infty$.

Proof

We consider $I_R(x) = \int_C e^{i\,x\,t^2}\, dt = \int_0^R e^{i\,x\,t^2}\, dt$ and deform the path C in the complex plane such that $I_R(x) = \int_{C_1} e^{i\,x\,t^2}\, dt + \int_{C_2} e^{i\,x\,t^2}\, dt$ (Figure 1.1).

By setting $t = s\, e^{i\,\pi/4}$ on C_1 and $t = R\,(\cos\theta + i\, \sin\theta)$ on C_2, we get

$$I_R(x) = e^{i\,\pi/4} \int_0^R e^{-x\,s^2}\, ds - \int_0^{\pi/4} e^{-x\,R^2\, \sin\theta}\, e^{-i\,x\,R^2\, \cos\theta}\, R\, d\theta \; . \tag{1.15}$$

Applying Cauchy inequality and Laplace's method to the second integral, we get

$$\left| \int_0^{\pi/4} e^{-x\,R^2\, \sin\theta}\, e^{-i\,x\,R^2\, \cos\theta}\, R\, d\theta \right| \le \frac{R\,\pi}{4} \int_0^{\pi/4} e^{-x\,R^2\, \sin\theta}\, d\theta \sim \frac{\pi}{4\,x\,R} \quad \text{for } x \to +\infty \; . \tag{1.16}$$

When $R \to \infty$, the first integral reads

$$e^{i\,\pi/4} \int_0^R e^{-x\,s^2}\, ds \to e^{i\,\pi/4} \int_0^\infty e^{-x\,s^2}\, ds = \frac{1}{\sqrt{2\,x}} e^{i\,\pi/4} \int_0^\infty e^{-u^2/2}\, ds = \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{i\,\pi/4} \; . \tag{1.17}$$

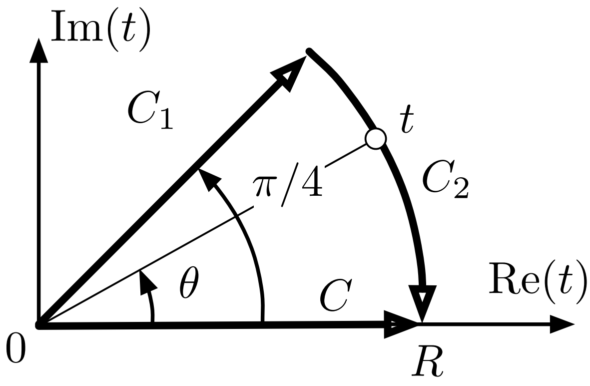

Figure 1.1: Complex plane

1.5.4 Stationnary phase method

Example 4, p. 280

Using the stationnary phase method, show that $I(x) = \int_{-\infty}^{\infty} \cos(x\,t^2 - t)dt \sim \sqrt{\frac{\pi}{2\,x}}$ for $x \to +\infty$.

Formula

We recall that $\int_a^b f(t)\, e^{i\,x\,\psi(t)}\,dt \sim f(a)\,\exp\left[i\,x\,\psi(a) + \epsilon\, i\,\frac{\pi}{2\,p}\right]\,\left[x\,\left|\psi^{(p)}(a)\right|\right]^{-1/p}\,(p!)^{1/p}\,\Gamma(1/p)/p$ for $x \to \infty$, with $\epsilon = \text{sign}\left[\psi^{(p)}(a)\right]$, when $\psi(t)$ can be approximated by $\psi(a) + \frac{1}{p!}\,(t-a)^p\,\psi^{(p)}(a)$ in the vicinity of it extremum supposed to be reached for $x = a$.

Physical applications

The stationnary phase method is mandatory to explain the concept of group velocity of wave packets. Given the 1D dispersion relation $\omega = \Omega(k)$ where ω is the pulsation and k in the wave superposition $u(x,t) = \int_{I\!R} f(k)\, e^{i\,[k\,x - \Omega(k)\,t]}\,dk$, the asymptotic behavior of $I(t) = u(v\,t, t)$ for $t \to \infty$ is derived with the stationnary phase method where $\psi(k) = k\,v - \Omega(k)$. Since the behavior of I only depends of the wave number k_v such that $\Omega'(k_v) = v$, the quantity $c_g(k) = \Omega'(k)$ is called the group velocity.

1.5.5 Steepest descent method (optional)

Example 1, p. 281 and Example 10 p. 296

Using the steepest descent method, show that:

(a) $I(x) = \int_0^1 \ln t\, e^{i\,x\,t}\,dt \sim -\frac{i\,\ln x}{x} - \frac{i\,\gamma + \pi/2}{x} + i\,e^{i\,x}\sum_{n=1}^{\infty}\frac{(-i)^n\,(n-1)!}{x^{n+1}}$ for $x \to +\infty$, where Euler's constant is $\gamma = -\int_0^{\infty} e^{-u}\,\ln u\,du = 0.5772...$

(b) $I(x) = \int_0^1 e^{-4\,x\,t^2}\,\cos(5\,x\,t - x\,t^3)\,dt \sim \frac{1}{2}\,e^{-x}\,\sqrt{\pi/x}$ for $x \to +\infty$

1.6 Perturbations Series

1.6.1 Perturbation of an eigenvalue problem (optional)

Section 7.3, p. 330 and Example 2, p. 334

We consider the Schrödinger equation $-y'' + \left(\frac{x^2}{4} + \epsilon\, x\right) y = E\, y$ subject to the boundary condition $\lim_{|x|\to\infty} y(x) = 0$. In the harmonic potential case $\epsilon = 0$, show that $E_0 = 1/2$ is an eigenvalue and $y_0(x) = e^{-x^2/4}$ an associated eigenfunction.

Using perturbation theory, show that $E = 1/2 - \epsilon^2 + O(\epsilon^3)$ for $\epsilon \to 0$. Recover that result through a translation for x.

Physical applications

In a lot of physical applications, one is often led to find the spectrum of a perturbed operator $A = A_0 + \epsilon A_1$, where ϵ is a small parameter, knowing the spectrum of A_0. Knowing an eigenvalue λ_0 of A_0 and the associated eigenvector y_0, the first order of the approximation $\lambda = \lambda_0 + \epsilon\, \lambda_1$ with $y = y_0 + \epsilon\, y_1$ leads to $(A_0 - \lambda_0)\, y_1 = -(A_1 - \lambda_0\,) y_1$. Defining the adjoint A^* of A by $< A\, y, z >=< y, A^*\, z >$ for all (y, z), where $< ., . >$ is a scalar product to be chosen, the compatibility condition of the first order equation leads to $\lambda_1 =< A_1\, y_0, y_0^* > / < y_0, y_0^* >$ where y_0^* is an eigenvector $A_0^*\, y_0^* = \lambda_0\, y_0^*$ of A_0^*, supposed to be known. The same procedure can be applied to go at higher order in ϵ.

1.7 Boundary Layer Theory

1.7.1 Boundary layer

Example 1, p. 419

We consider the differential equation $\epsilon\, y'' + (1 + \epsilon)\, y' + y = 0$ with the boundary conditions $y(0) = 0$ and $y(1) = 1$. Find the exact solution.

For $\epsilon \to 0$, show that the outer solution satisfies $y'_{out} + y_{out} = 0$ with $y_{out}(0) = 0$ and the inner solution is such that $Y''_{in} + Y_{in} = 0$ with $Y_{in}(0) = 0$ and $Y_{in}(+\infty) = e$ with $y_{in}(x) = Y_{in}(X)$ and $x = \epsilon\, X$. Draw the matching between the inner and outer solutions. Write a uniform approximation under the form $y_{unif} = y_{in} + y_{out} + y_{match}$.

Physical applications

The viscosity of fluid flows, when it is small, only act in small boundary layers close to walls. In that case, the fluid can be consider as perfect (outer solution) and matching with viscous profiles (inner solution) close to the walls.

1.8 WKB Theory

1.8.1 Schrödinger equation

Example 1, p. 486

We consider the Schrödinger equation $\epsilon^2\, y'' = Q(x)\, y$ with $Q(x) \neq 0$ for the x considered. Show that

$$y(x) \sim c_1\, Q^{-1/4}(x)\, \exp\left(\frac{1}{\epsilon}\int_a^x \sqrt{Q(t)}\, dt\right) + c_2\, Q^{-1/4}(x)\, \exp\left(-\frac{1}{\epsilon}\int_a^x \sqrt{Q(t)}\, dt\right)\,, \qquad (1.18)$$

provided $\epsilon\, S_1 \ll S_0$, $\epsilon\, S_2 \ll S_1$ and $\epsilon\, S_2 \ll 1$ for $\epsilon \to 0$ with

$$S_0(x) = \int^x \sqrt{Q(t)}\, dt, \qquad S_1(x) = -\frac{1}{4}\ln Q(x) \qquad S_2(x) = \int^x \left[\frac{Q''}{8\, Q^{3/2}} - \frac{5\,(Q')^2}{32\, Q^{5/2}}\right] dt\,. \tag{1.19}$$

Physical applications

The WKB approach is widely used in optics. The first order describes the ray tracing of the electromagnetic waves and is called the "geometrical optics approximation". The second ordre describes the amplitude of the electromagnetic waves and is calle the "physical optics approximation". The small parameter ϵ is related to the wave number that is suppose to be small compared the variation scale of $\sqrt{Q}$, relatif the refraction index. The integral in the approximation related to the optical path.

1.8.2 Sturm-Liouville problem

Example 5, p. 490

We consider the Sturm-Liouville problem $y'' + E\,(x+\pi)^4\, y = 0$ with the boundary conditions $y(0) = y(\pi) = 0$. Using the WKB theory, show that

$$E_n \sim \frac{9\,n^2}{49\,\pi^4} \quad \text{and} \quad y_n(x) \sim \sqrt{\frac{6}{7\,\pi^3}}\, \frac{\sin\left[n\,\left(x^3 + 3\,x^2\,\pi + 3\,\pi^2\,x\right)/(7\,\pi^2)\right]}{(\pi + x)} \quad \text{for } n \to \infty\,, \tag{1.20}$$

are the eigenvalues and eigenvectors at large n with the normalization $\int_0^\pi y_n^2\,(x+\pi)^4\, dx = 1$.

1.8.3 Airy equation

Example 1, p. 494

Using the WKB theory, show the solution of the Airy equations satisfies

$$y(x) \sim c_\pm\, x^{-1/4}\, e^{\pm 2\, x^{3/2}/3} \left(1 \pm \frac{5}{48}\, x^{-3/2}\right)\,, \tag{1.21}$$

for $x \to \infty$, where c_+ and c_- are arbitrary constants.

1.8.4 Turning point (optional)

Section 10.4, p. 504

We consider the Schrödinger equation $\epsilon^2\, y'' = Q(x)\, y$ with boundary condition $y(+\infty) = 0$ such that $Q(x) > 0$ for $x > 0$, $Q(x) < 0$ for $x < 0$ and $Q(x) \sim a\, x$ for $x \to 0$ with $a > 0$. We assume that $Q(x) \ll x^{-2}$ for $x \to \pm\infty$. Show the approximations for $\epsilon \to 0$:

$$\begin{aligned}
y_{III}(x) &= 2\,C\,[-Q(x)]^{-1/4}\, \sin\left[\frac{1}{\epsilon}\int_x^0 \sqrt{-Q(t)}\, dt + \pi/4\right] && \text{for } x \ll -\epsilon^{2/3}\,,\\
y_{II}(x) &= 2\,C\,\sqrt{\pi}\,(a\,\epsilon)^{-1/6}\, \mathrm{Ai}\left(\epsilon^{-2/3}\, a^{1/3}\, x\right) && \text{for } |x| \ll 1\,,\\
y_I(x) &= C\,[Q(x)]^{-1/4}\, \exp\left[-\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\, dt\right] && \text{for } x \gg \epsilon^{2/3}\,,
\end{aligned}$$

where C is an arbitrary constant. Draw schematically the solution.

Physical applications

Turning points describe the transition between oscillatory and expontial behavior of physical waves. Matching these two behavior is an important problem. Intersesting phenomena such as the tunnel effect in quantum mechanics can be described with this approach.

1.9 Multi-Scale Analysis

1.9.1 Periodic solutions

Section 11.2, p. 549, Example 2, p. 554

We use the multiscale analysis $y(t) \sim Y_0(t, \epsilon t) + \epsilon\, Y_1(t, \epsilon t)$ for $\epsilon \to 0$ with:

$Y_0(t,\tau) = R(\tau)\left[e^{i\,\theta(\tau)}\, e^{i\,t} + e^{-i\,\theta(\tau)}\, e^{-i\,t}\right]$. Show the following relations:

(a) Duffing equation: $y'' + y + \epsilon\, y^3 = 0$: $\frac{dR}{d\tau} = 0$ and $\frac{d\theta}{d\tau} = 3\, R^2/2$.

(b) Rayleigh oscillator: $y'' + y = \epsilon\, \left[y' - \frac{1}{3}\,(y')^3\right]$: $2\,\frac{dR}{d\tau} = R - R^3$ and $\frac{d\theta}{d\tau} = 0$.

Draw the shape of these solutions.

Physical applications

Amplitude equations are used to describe the nonlinear saturation of an instability, closed to its threshold. The multi-scale analysis is one of several tools (e.g. normal forms) used to derive these amplitude equations.

1.9.2 Mathieu equation (optional)

Section 11.4, p. 560

We use the multiscale analysis $y(t) \sim Y_0(t, \epsilon t)$ for $\epsilon \to 0$ with $Y_0(t,\tau) = A(\tau) e^{i\,\frac{t}{2}} + A^*(\tau) e^{-i\,\theta(\tau)}\, e^{-i\,\frac{t}{2}}$ for the Mathieu equation $y'' + \left[\frac{1}{4} + (a_1 + 2\,\cos t)\,\epsilon\right] y = 0$. Show that $|a_1| = 1$ is the limit for the stability of the equilibrium $y = 0$.

Physical applications

The parametric instability is encountered when a parameter of an oscillating system is forced at a frequency close to twice its natural oscillating frequency. This is the case of a pendulum which length is oscillating. x

Chapter 2

Training problems

The following problems are designed to be made without documents during two or three hours.

2.1 First four chapters of the book

2.1.1 Frobenius expansions

We recall that if $\mathcal{L}(y) = x^2\, y'' + x\, p(x)\, y' + q(x)\, y$ with $p(x) = \sum_{n=0}^{\infty} p_n\, x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n\, x^n$, for $y = x^\alpha \sum_{n=0}^{\infty} a_n\, x^n$ we have:

$$\mathcal{L}(y) = x^\alpha\, P(\alpha)\, a_0 + x^\alpha \sum_{n=1}^{\infty} \left[P(n+\alpha)\, a_n + \sum_{k=1}^{n} (n-k+\alpha)\, p_k\, a_{n-k} + \sum_{k=1}^{n} q_k\, a_{n-k} \right] x^n$$

with $P(X) = X\,(X-1) + p_0\, X + q_0 = X^2 + (p_0 - 1)\, X + q_0$.

1) Use the Frobenius method to solve the Bessel equation $x^2\, y'' + x\, y' + (x^2 - \nu^2)\, y = 0$ in the case $2\,\nu \notin \mathbb{N}$.

2.1.2 Watson Lemma and stationnary phase method

2) Find an equivalent of $I(x) = \int_0^1 \cos(x\, \cos t)\, dt$ for $x \to +\infty$.

3) We consider $u(x,t) = \int_0^\infty \cos\left[k\, x - \Omega(k)\, t\right] dk$ with $\Omega(k) = \sqrt{k}$ and define $I(t) = u(t/2, t)$. Show that $I(t) \sim \alpha\, \cos(t/2 + \beta)/\sqrt{t}$ for $x \to +\infty$ and express the constants α and β.

2.1.3 Dominant balance

4) Show that the solutions of $y'' = \sqrt{x}\, y$ for $x > 0$ satisfy either $y_1 \sim c_1\, x^{\alpha_1} \exp(\gamma_1\, x^{\beta_1})$ or $y_2 \sim c_2\, x^{\alpha_2} \exp(\gamma_2\, x^{\beta_2})$ for $x \to +\infty$ and give the values of the six constants $(\alpha_i, \beta_i, \gamma_i)$.

2.1.4 Critical points

We consider the dynamical system $\dot{x} = \sin x\, \cos y$, $\dot{y} = -\cos x\, \sin y$ for $(x,y) \in [0,\pi]^2$.

5) Show that there are five equilibria and compute them.

6) Study the stability of these equilibria.
7) Draw the trajectories in the (x, y) definition square.

2.1.5 Steepest descent

8) By deforming the integration path in the complex plane and using the Laplace method, show that $I(x) = \int_0^\infty e^{-i\,x\,t^2}\,dt \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\,e^{-i\,\pi/4}$ for $x \to +\infty$.

Corrigé

1)One finds the two basic solutions:

$$y_\pm(x) = a_0\,\Gamma(\nu+1)\,x^{\pm\nu}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\,x\right)^{2\,n}}{n!\,\Gamma(\pm\nu+n+1)}$$

2)Since $I(x) = \int_a^b f(t)\,e^{i\,x\,\psi(t)}\,dt \sim f(a)\,\exp\left[i\,x\,\psi(a) + \epsilon\,i\,\frac{\pi}{2\,p}\right]\,\left[x\,\left|\psi^{(p)}(a)\right|\right]^{-1/p}\,(p!)^{1/p}\,\Gamma(1/p)/p$ for $x \to \infty$, with $\epsilon = \text{sign}\left[\psi^{(p)}(a)\right]$, when $\phi(t)$ can be approximated by $\phi(a) + \frac{1}{p!}\,(t-a)^p\,\phi^{(p)}(a)$, in the vicinity of it extremum.

In the case $p = 2$, $I(x) \sim f(a)\,\exp\left[i\,x\,\psi(a) + \epsilon\,i\,\frac{\pi}{4}\right]\,\left[x\,|\psi''(a)|\right]^{-1/2}\,\sqrt{\pi/2}$ for $x \to \infty$, with $\epsilon = \text{sign}\,[\psi''(a)]$, when $\phi(t)$ can be approximated by $\phi(a) + \frac{1}{2}(t-a)^2\,\phi''(a)$.

Here we have $I(x) = \text{Re}\left[\int_0^1 f(t)\,e^{i\,x\,\psi(t)}\,dt\right]$ with $f(t) = 1$ and $\psi(t) = \cos(t) = 1 - t^2/2 + O(t^4)$. Since $\psi(0) = 1$ and $\psi''(0) = -1$, we have $I(x) \sim \text{Re}\left(e^{ix - i\,\pi/4}\,\sqrt{\frac{\pi}{2\,x}}\right) = \cos(x - \pi/4)\,\sqrt{\frac{\pi}{2\,x}}$. **3)**We consider $J(t) = \int_0^1 f(k)\,e^{i\,t\psi(k)}\,dk$ with $f(k) = 1$ and $\psi(k) = k/2 - \Omega(k)$. This phase is extremum for $\psi'(k) = 1/2 - \Omega'(k) = (1 - 1/\sqrt{k})/2$, that is for $k_* = 1$. Since $\psi''(1) = 1/4$, we have $J(t) \sim e^{-i\,t/2 + i\,\pi/4}\,\sqrt{8\,\pi/t}$ for $t \to +\infty$. Thus, $I(t) = \text{Re}\,[J(t)] \sim 2\sqrt{2\,\pi/x}\,\cos(t/2 - \pi/4)$. We have $\alpha = 2\,\sqrt{2\,\pi}$ and $\beta = -\pi/4$.

4)Let $y = \exp(S_a + C)$ with $C \ll S_a$ for $x \to \infty$. The equation reads $S_a'' + C'' + (S_a' + C')^2 = \sqrt{x}$. Assuming $S_a'' \ll (S_a')^2$, we get $S_a' = \pm x^{1/4}$ leading to $S_a = \pm(4/5)\,x^{5/4} + cste$ and $S_a'' = \pm(1/4)\,x^{-3/4}$. The assumption $S_a'' \ll (S_a')^2$ is thus valid. The equation now reads $\pm(1/4)\,x^{-3/4} + C'' \pm 2\,x^{1/4}C' + C'^2 = 0$. Assuming $C'^2 \ll x^{1/4}\,C'$ and $C'' \ll x^{1/4}\,C'$, we have $C' = -(1/8)x^{-1}$ leading to $C = -(1/8)\,\text{Ln}\,(x) + cste$. The two assumptions on C are satisfied. Thus, $\alpha_1 = \alpha_2 = -1/8$, $\gamma_1 = -\gamma_2 = 4/5$ and $\beta_1 = \beta_2 = 5/4$.

5)The equilibria are $A = (0, 0)$, $B = (0, 2\pi)$, $C = (\pi, \pi)$, $D = (\pi, 0)$ and $I = (\pi/2, \pi/2)$.

6)The jacobian matrix reads $J_{11} = \cos x\ \cos y$, $J_{12} = -\sin x\ \sin y$, $J_{21} = \sin x\ \sin y$ and $J_{22} = -\cos x\ \cos y$. I is marginal while A, B, C and D are unstable.

7)The trajectories are circles close to I and quasi-squares close to $ABCD$.

8)Same as in the course but with and angle $-\pi/4$ instead of $\pi/4$ for the complex integration path.

2.2 Exam given in 2016

2.2.1 Ordinary Differential Equations and Green function

We consider the ordinary differential equation $y'' - y = f$ for $x \in \mathbb{R}$ with $f(x) = 1/\cosh(x)$ and the boundary conditions $y(\pm\infty) = 0$.

1) Use the variation of parameters to solve this problem.
2) Use the Green function method to solve this problem and compare.

2.2.2 Frobenius expansions

We recall that if $\mathcal{L}(y) = x^2\, y'' + x\, p(x)\, y' + q(x)\, y$ with $p(x) = \sum_{n=0}^{\infty} p_n\, x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n\, x^n$ we have

$$\mathcal{L}\left(x^\alpha \sum_{n=0}^{\infty} a_n\, x^n\right) = x^\alpha\, P(\alpha)\, a_0 + x^\alpha \sum_{n=1}^{\infty} \left[P(n+\alpha)\, a_n + \sum_{k=1}^{n} (n-k+\alpha)\, p_k\, a_{n-k} + \sum_{k=1}^{n} q_k\, a_{n-k}\right] x^n$$

with $P(X) = X\,(X-1) + p_0\, X + q_0 = X^2 + (p_0 - 1)\, X + q_0$.

We consider the modified Bessel equation $\mathcal{L}(y) = x^2\, y'' + x\, y' - x^2\, y = 0$. We define the functions $y(x,\alpha) = x^\alpha \sum_{n=0}^{\infty} a_n(\alpha)\, x^n$ for $\alpha \in \mathbb{C}$ where a_0 is a given constant, $a_1 = 0$ and $a_{n+2}(\alpha) = a_n(\alpha)/(\alpha+n)^2$ for $n \in \mathbb{N}$.

3) Show that $I_0(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}x\right)^{2n}/(n!)^2$ is a solution of the given modified Bessel equation.
4) Show that $\frac{d}{d\alpha} a_{2p}(0)/a_{2p}(0) = -\, u_p$ where $u_p = \sum_{k=1}^{p} k^\beta$ where β is a constant and give the value of β. Deduce that $\frac{\partial}{\partial\alpha} y(x,0) = a_0\, \mathrm{Ln}\,(x)\, I_0(x) - a_0 \sum_{n=1}^{\infty} u_n\, \left(\frac{1}{2}x\right)^{2n}/(n!)^2$.
5) We define the function $K_0(x) = -[\mathrm{Ln}\,(\frac{1}{2}x) + \gamma]\, I_0(x) + \sum_{n=1}^{\infty} v_n\, \left(\frac{1}{2}x\right)^{2n}/(n!)^2$ where γ is the Euler constant ($\gamma \sim 0.5772$) and $v_n = \sum_{k=1}^{n}(1/k)$. Show that $K_0(x)$ is a solution of the given modified Bessel equation and is independant of $I_0(x)$.

2.2.3 Irregular singular points

6) Show that the solutions of $y'' = y\,\sqrt{x}$ for $x > 0$ satisfy either $y_1 \sim c_1\, x^{\alpha_1}\, \exp(\gamma_1\, x^{\beta_1})$ or $y_2 \sim c_2\, x^{\alpha_2}\, \exp(\gamma_2\, x^{\beta_2})$ for $x \to +\infty$ and give the values of the six constants $(\alpha_i, \beta_i, \gamma_i)$.

2.2.4 Laplace method

We recall that $\int_a^b f(t)\, e^{x\,\phi(t)}\, dt \sim 2\, f(c)\, e^{x\,\phi(c)}\, [-x\, \phi^{(p)}(c)]^{-1/p}\, \Gamma(1/p)\, (p!)^{1/p}/p$ for $x \to \infty$ when $\phi(t)$ can be approximated, in the vicinity of it maximum, by $\phi(c) + \frac{1}{p!}\,(t-c)^p\, \phi^{(p)}(c)$ if it is reached for $c \in]a,b[$. Since $\phi(c)$ is a maximum, we have p even and $\phi^{(p)}(c) < 0$.

7) The Gamma function is defined by $\Gamma(x) = \int_0^{\infty} e^{-u}\, u^{x-1}\, du$ for $x > 0$. Using the change of variable $u = x\,t$ and the Laplace method, show the Stirling's formula $\Gamma(x) \sim \alpha\, x^x\, e^{-x}/\sqrt{x}$ for $x \to +\infty$ and give the value of the constant α.

2.2.5 Stationnary phase method

We recall that $\int_a^b f(t)\, e^{i\,x\,\psi(t)}\, dt \sim f(a) \exp\left[i\,x\,\psi(a) + \epsilon\, i\, \frac{\pi}{2p}\right] \left[x\, \left|\psi^{(p)}(a)\right|\right]^{-1/p} (p!)^{1/p}\, \Gamma(1/p)/p$ for $x \to \infty$, with $\epsilon = \text{sign}\left[\psi^{(p)}(a)\right]$, when $\psi(t)$ can be approximated by $\psi(a) + \frac{1}{p!}\,(t-a)^p\, \psi^{(p)}(a)$ in the vicinity of it extremum supposed to be reached for $x = a$.

8) Find an equivalent of $I(x) = \int_0^1 \cos(x\, \cos t)\, dt$ for $x \to +\infty$.

2.2.6 Critical point

We consider the dynamical system $\dot{x} = \frac{1}{2}x - \frac{1}{2}x^3$ for $x(t) \in I\!R$ with the initial condition $x(0) = x_0$.

9) Show that there are three equilibria and compute them.
10) Study the stability of these equilibria.
11) Draw the trajectories in $I\!R$.
12) Draw, schematically, the graph of $x(t)$ as a function of t for various initial conditions $x_0 \in I\!R$.

2.2.7 Multi-scale analysis

We consider the Van der Pol oscillator $y'' + \epsilon\,(y^2-1)\,y' + y = 0$ where $y(t)$ is determined by the initial condition $y(0) = y_0$ and $y'(0) = v_0$. We use the multiscale analysis $y(t) = Y(t, \epsilon\, t)$ for $\epsilon \to 0$ with $Y(t,\tau) = Y_0(t,\tau) + \epsilon\, Y_1(t,\tau) + O(\epsilon^2)$.

13) Show that $Y_0(t,\tau) = A(\tau)\, e^{i\,t} + A^*(\tau)\, e^{-i\,t}$ where $A(\tau) = R(\tau)\, e^{i\,\theta(\tau)}$.
14) Write an equation $\frac{dA}{d\tau} = \alpha\, A\,(1 - |A|^2)$ to eliminate the secular term of the equation for Y_1 at the second order of the asymptotic expansion and give the values of the real constant α.
15) Draw the graph of the fonction $R(\tau)$ for $R(0) < 1$ and then for $R(0) > 1$.
16) Write the expression of the asymptotic solution $y(t)$ using the function R at the dominant order (Y_0 only) and draw its graph as a function of t. Express $R(0)$ and $\theta(0)$ as functions of y_0 and v_0.
17) Draw the trajectories in the (y, y') plane.

2.2.8 Boundary-layer theory

We consider the equation $\epsilon\, y'' + a(x)\, y' + b(x)\, y = 0$ with $a(x) = x^2 + 1$ and $b(x) = -x^3$ for $x \in [0,1]$ with $y(0) = y(1) = 1$ in the limit $\epsilon \to 0$.

The boundary-layer method for this type of equations consists in finding an outer solution $y_{out}(x)$ valid on the whole interval, excepted in a "boundary layer" whose size tends to zero when $\epsilon \to 0$.

If $a(x) > 0$, as this is the case for this example, we admit that the boundary layer develops in the vicinity of $x = 0$. In this layer, the inner solution is obtained by setting $y_{in}(x) = Y_{in}(X)$ with $x = \epsilon\, X$ with the following boundary conditions: $Y_{in}(0) = y(0)$ and $\lim_{X\to\infty} Y_{in}(X) = y_{out}(0)$.

18) Compute the outer solution $y_{out}(x)$.
19) Compute the inner solution $y_{in}(x)$.
20) Show that exists a constant y_{match} such that $y_{unif}(x) = y_{out}(x) + y_{in}(x) - y_{match}$ is a uniform approximation. Give the final expression of $y_{unif}(x)$. Draw schematically $y(x)$ for a small ϵ.

Corrigé

1)Using the notations of the course, we have $u_1' = -f\,y_2/W = e^{-x}/(e^x + e^{-x})$ and $u_2' = f\,y_1/W = -e^x/(e^x + e^{-x})$. Changing into the variable $s = e^{-x}$ for u_1 and $s = e^x$ for u_2, one gets $u_1 = -\int^{e^{-x}} \frac{s}{s^2+1}\,ds = -\mathrm{Ln}\,\sqrt{e^{-2x}+1} + C_1$ and $u_2 = -\int^{e^x} \frac{s}{s^2+1}\,ds = -\mathrm{Ln}\,\sqrt{e^{2x}+1} + C_2$. The boundary conditions lead to $C_1 = C_2 = 0$ and one can check that the solution $y = -e^x\,\mathrm{Ln}\,\sqrt{e^{-2x}+1} - e^{-x}\,\mathrm{Ln}\,\sqrt{e^{2x}+1}$ satisfies $y(\pm\infty) = 0$. **2)**The Green function is $G(x) = e^{-|x|}/2$ and $y = \int_{\mathbb{R}} f(s)\,G(x-s)\,ds$ reads $y(x) = -\int_{-\infty}^{x} \frac{e^{s-x}}{e^s+e^{-s}}\,ds - \int_x^{\infty} \frac{e^{-s+x}}{e^s+e^{-s}}\,ds$. This leads to the previously found solution.

3)Since $p_0 = 1$ and $q_0 = \nu^2$ for the modified Bessel equation, we have $P(X) = X^2 - \nu^2$. For this equation, we have

$$\mathcal{L}\left(x^\alpha \sum_{n=0}^{\infty} a_n\,x^n\right) = x^\alpha\,P(\alpha)\,a_0 + x^\alpha\,P(\alpha+1)\,a_1\,x + x^\alpha \sum_{n=2}^{\infty} \left\{[(n+\alpha)^2 - \nu^2]\,a_n - a_{n-2}\right\}\,x^n\;.$$

For $\nu = 0$, $X = 0$ is a double root. For this case, we note that $\mathcal{L}[y(x,\alpha)] = x^\alpha\,P(\alpha)$. The function $y(x,0) = \sum_{n=0}^{\infty} a_n(\alpha)\,x^n$ with $a_{2p} = (2^p\,p!)^{-2}\,a_0$ for $n = 2p$ even and $a_{2p+1} = 0$ for $n = 2p+1$ even is solution of equations. $I_0(x)$ is such a solution with $a_0 = 1$. **4)**One can compute that $a_{2p}(\alpha) = a_0 \prod_{k=1}^{p} 1/(\alpha+2\,k)^2$ for $n = 2p$ even and $a_{2p+1}(\alpha) = 0$ for $n = 2p+1$ even. We thus have $a_{2p}'(\alpha)/a_{2p}(\alpha) = -2\sum_{k=1}^{p} 1/(\alpha+2\,k)$ and thus $a_{2p}'(0) = -a_{2p}(0)\,\sum_{k=1}^{p} 1/k$. Thus $\frac{\partial}{\partial\alpha}y(x,0) = a_0\,\mathrm{Ln}\,x\,I_0(x) - a_0\,\sum_{n=1}^{\infty} u_n\,\left(\frac{1}{2}x\right)^{2n}/(n!)^2$ with $u_n = \sum_{k=1}^{n} 1/k$. **5)**We note that $\mathcal{L}[\frac{\partial}{\partial\alpha}y(x,\alpha)] = P(\alpha)\,x^\alpha\,\mathrm{Ln}\,x + P'(\alpha)\,x^\alpha$. Since $P(0) = P'(0) = 0$, we have $\mathcal{L}[\frac{\partial}{\partial\alpha}y(x,0)] = 0$. Thus Choosing $a_0 = -1$, one has $K_0(x) = \frac{\partial}{\partial\alpha}y(x,0) + (\mathrm{Ln}\,2 - \gamma)\,I_0(x)$. This solution is independant of $I_0(x)$ since $\frac{\partial}{\partial\alpha}y(x,0)$ and $I_0(x)$ are independant.

6)We suppose that $y = \exp(S_a + C)$ with $C \ll S_a$ for $x \to +\infty$. The equation reads $(S_a'' + C'') + (S_a' + C')^2 = x^{1/2}$ and is approximated, at the dominant order, by $S_a'' + (S_a')^2 = x^{1/2}$. Assuming that $S_a'' \ll (S_a')^2$, we have $S_a' = \pm x^{1/4}$ and $S_a'' = \pm\frac{1}{4}\,x^{-3/4}$ is indeed smaller than $(S_a')^2 = x^{1/2}$ for $x \to \infty$. The next order reads $S_a'' + C'' + 2\,S_a'\,C' + (C')^2 = \pm\frac{1}{4}\,x^{-3/4} + C'' \pm 2\,x^{1/4}\,C' + (C')^2 = 0$. Assuming $C'' \ll x^{-3/4}$ and $(C')^2 \ll x^{-3/4}$, one gets $C' = -\frac{1}{8}\,x^{-1}$, which satisfies theses assumptions. Since $S_a = \pm\frac{4}{5}\,x^{5/4} + \mathit{constant}$ and $C = -\frac{1}{8}\mathrm{Ln}\,x$, we have $y_1 \sim c_1\,x^{-1/8}\exp(\frac{4}{5}\,x^{5/4})$ and $y_2 \sim c_2\,x^{-1/8}\exp(-\frac{4}{5}\,x^{5/4})$. Thus, $\alpha_1 = \alpha_2 = -1/8$, $\beta_1 = \beta_2 = 5/4$ and $\gamma_1 = -\gamma_2 = 4/5$.

7)The change of variable $u = x\,t$ leads to $\Gamma(x) = x\,\int_0^{\infty} \exp[-x\,t + (x-1)\,\mathrm{Ln}\,(xt)]\,dt$. This can be written under the form $\Gamma(x) = x^x\,\int_0^{\infty} \frac{1}{t}\exp[x\,\phi(t)]\,dt$ with $\phi(t) = \mathrm{Ln}\,t - t$. Since $\phi'(t) = 1/t - 1$ and $\phi''(t) = -1/t^2$, $\phi(t)$ is maximum for $t = 1$ with $\phi(1) = -1$ and $\phi''(1) = -1$. Applying the given formula for $p = 2$, one get $\Gamma(x) \sim x^x\,e^{-x}\sqrt{2\,\pi/x}$. Thus $\alpha = \sqrt{2\,\pi}$.

8)We have $I(x) = \mathrm{Re}\left[\int_0^1 f(t)\,e^{i\,x\,\psi(t)}\,dt\right]$ with $f(t) = 1$ and $\psi(t) = \cos(t) = 1 - t^2/2 + O(t^4)$. Since $\psi(0) = 1$ and $\psi''(0) = -1$, we have $I(x) \sim \mathrm{Re}\left(e^{ix - i\,\pi/4}\sqrt{\frac{\pi}{2\,x}}\right) = \cos(x - \pi/4)\,\sqrt{\frac{\pi}{2\,x}}$.

9)The equilibria are $x_e \in \{-1, 0, 1\}$. **10)**The stability of the equilibria of $\dot{x} = f(x)$ depend on the signe of $f'(x_e) = \frac{1}{2} - \frac{3}{2}\,x_e^2$. The equilibrium $x_e = 0$ is stable since $f'(0) = 1/2 > 0$. The equilibria $x_e = \pm 1$ are stable since $f'(\pm 1) = -1 < 0$. **11)**The trajectoires are : $]-\infty, 1[$ with increasing x, $\{-1\}$, $]-1, 0[$ with decreasing x, $\{0\}$, $]0, 1[$ with increasing x, $\{1\}$ and $]1, \infty[$ with decreasing x. **12)**For $x_0 \in]0, 1[$, $x(t)$ is increasingly converging towards 1 for large t, $x(t) = 1$ is constant for $x_0 = 1$, for $x_0 \in]1, \infty[$, $x(t)$ is decreasly converging towards 1. The symetry $x \to -x$ leads to the behavior or the other curves.

13) We have $y'(t) = \frac{\partial Y}{\partial t}(t, \epsilon\,t) + \epsilon\,\frac{\partial Y}{\partial \tau}(t, \epsilon\,t) = \frac{\partial Y_0}{\partial t}(t, \epsilon\,t) + O(\epsilon)$ and $y''(t) = \frac{\partial^2 Y}{\partial t^2} + 2\,\epsilon\,\frac{\partial^2 Y}{\partial t\,\partial\tau} + \epsilon^2\,\frac{\partial^2 Y}{\partial \tau^2} =$

$\frac{\partial^2 Y_0}{\partial t^2}+\epsilon\left[\frac{\partial^2 Y_1}{\partial t^2}+2\,\frac{\partial^2 Y_0}{\partial t\,\partial\tau}\right]+O(\epsilon^2)$. At the dominant order, the equation reads $\frac{\partial^2 Y_0}{\partial t^2}+Y_0=0$, leading to $Y_0(t,\tau)=A(\tau)\,e^{i\,t}+A^*(\tau)\,e^{-i\,t}$. At the next order, one gets $\frac{\partial^2 Y_1}{\partial t^2}+Y_1=-2\,\frac{\partial^2 Y_0}{\partial t\,\partial\tau}+\frac{\partial Y_0}{\partial t}(Y_0^2-1)=-2(i\,\frac{dA}{d\tau}\,e^{it}-i\,\frac{dA^*}{d\tau}\,e^{-it})+(i\,A\,e^{it}-i\,A^*\,e^{-it})(1-|A|^2+A^2\,e^{2it}+A^{*2}\,e^{-2it})=F_1(A,A^*)\,e^{it}+F_{-1}(A,A^*)\,e^{-it}+F_3(A,A^*)\,e^{3it}+F_{-3}(A,A^*)\,e^{-3it}$ with $F_1(A,A^*)=i\,(-2\frac{dA}{d\tau}+A-|A|^2\,A)$. We impose $F_1(A,A^*)=0$ to eliminate the secular term, leading to $\frac{dA}{d\tau}=\frac{1}{2}(A-|A|^2\,A)$. Thus, $\alpha=1/2$. **14)**We have $\frac{dR}{d\tau}=\frac{1}{2}R-\frac{1}{2}R^3$. For $R(0)\in]0,1[$, $R(\tau)$ is increasingly converging towards 1 for large τ. For $R(0)\in]1,\infty[$, $R(\tau)$ is decreasingly converging towards 1 for large τ. **15)**The graph of $R(\tau)$ have been treated in a previous question. **16)**Since $\frac{d\theta}{d\tau}=0$ leads to $\theta(\tau)=\theta(0)$, we have obtained $y(t)=2\,R(\epsilon\,t)\cos[t+\theta(0)]+O(\epsilon)$. The initial conditions leads to $R(0)=\frac{1}{2}\,\sqrt{y_0^2+v_0^2}$ and $\theta(0)=-\mathrm{atan}\,(v_0/y_0)$. **17)**Trajectories are spirals converging towards the limit cycle.

18) Setting ϵ in the equation, we get $y'_{out}/y_{out}=-b/a=x^3/(x^2+1)=x-x/(x^2+1)$. Thus $y_{out}(x)=A\,\exp(x^2/2)/\sqrt{x^2+1}$. The boundary condition $y_{out}(1)=1$ leads to $A=\sqrt{2/e}$. Thus $y_{out}(x)=\sqrt{2/e}\,\exp(x^2/2)/\sqrt{x^2+1}$. **19)**Setting $y_{in}(x)=Y_{in}(x/\epsilon)$ leads to $Y''_{in}(X)+Y_{in}(X)=0$ at the leading order. The solution $Y_{in}(X)=B\,\exp(-X)+C$ must satisfy the boundary conditions $Y_{in}(0)=1$ and $\lim_{X\to\infty}Y_{in}(X)=y_{out}(0)=\sqrt{2/e}$. We thus have $B=1-\sqrt{2/e}$ and $C=\sqrt{2/e}$. Thus $y_{in}(x)=(1-\sqrt{2/e})\,\exp(-x/\epsilon)+\sqrt{2/e}$ **20)**Setting $y_{match}=-\sqrt{2/e}$, we obtain the uniform approximation $y_{unif}(x)=(1-\sqrt{2/e})\,\exp(-x/\epsilon)+\sqrt{2/e}\,\exp(x^2/2)/\sqrt{x^2+1}$. The graph of $y(x)$ is made of a rapidly decreasing curve from 1 to $y(0^+)=\sqrt{2/e}$ in the vicinity of $x=0$ followed by an increasing curve up to $y(1)=1$.